Two Multimessages Capacity of Ring Networks

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Outline

- Routing rate region in networks
- The inequality elimination technique
- Routing capacities of ring networks
- PdE bound and their strengthening in ring network
- General multiple multicast routing in ring networks

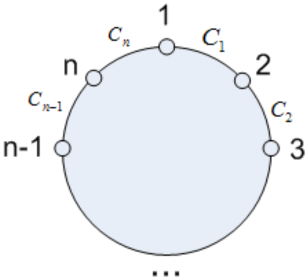


Figure: An undirected ring network

Routing Rate Region

Approach 1: Linear Programming

Approach 2: Dual problem, Japanese Theorem

- Assign weight $W_e$to edge C_e
- For session A_i:
  - Let $l_i$be the minimum tree length of routing spanning trees
  - Let R_{A_i}be the rate of it

Theorem

(Extended Japanese Theorem) Rate bounds $\sum_i l_i R_{A_i} \leq \sum_{e \in E} W_e C_e$*characterize the routing rate region*

Japanese theorem is an infinite set of linear inequalities. The elimination techniques is a way to check the redundancies:

Theorem

(Inequality elimination) Let $W = (W_1, W_2, \cdots, W_m)$ *and* $W' = (W'_1, W'_2, \cdots, W'_m)$ *be two vectors of edge distances satisfying:*

1. for every i, $W'_i = 0$ *implies* $W_i = 0$,

2. for every session A_i, *if* T_i*is the collection of shortest trees by* W, *and* T'_i*is the collection of shortest trees by* W', *then* $T'_i \subset T_i$,

then the inequality corresponding to W' *is redundant in the presence of the inequality corresponding to* W_i.

Routing Capacity of Ring Networks

To find the capacity region, one can use Inequality Elimination theorem to find the minimal set of edge distances vectors that describe the routing capacity. Since the general multiple multicast problem is very challenging, here we consider two special cases.

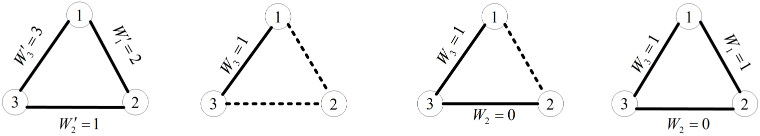
Undirected Rings with Line Sessions

A line session has the source and all destinations of the session as successive nodes on the ring.
We propose an algorithm that for given $W' = (W'_1, \cdots, W'_n)$ finds the binary vector $W = (W_1, \cdots, W_n)$ such that W and W' satisfy Inequality Elimination theorem conditions.
Algorithm1:

1. Set $W_j = 0$if $W'_j = 0$ and set $i = 1$
2. Complete the following steps:
   2.1 Set $W_i = 1$
   2.2 Search for an index j such that $\sum_{k=i}^{i+j-1} W'_k < \max W'$, but $\sum_{k=i}^{i+j} W'_k \geq \max W'$. If such a j exists, it must be unique. In this case increase i by j and return to Step 1. If no such j exists, go on to Step 3.
3. Set the remaining edge distances in W' to zero.

Example

We show how algorithm works for $W' = (2, 1, 3)$ where we choose the counterclockwise direction



Undirected Rings with Unicast and Broadcast Sessions

Result: Only need binary edge weights

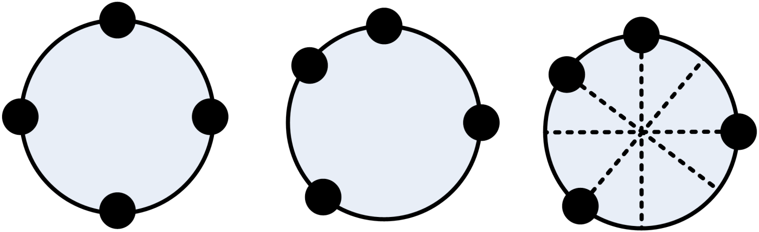
Procedure to go from arbitrary edge weights to binary edge weights with the same minimal length routing paths and trees involves the following Basic Generation Method and some modifications in some special cases:

Algorithm2:

1. If all elements of W' are equal, set W to all ones vector.
2. Draw a circle with nodes on it corresponding to the nodes of the network and with length of arcs proportional to the length of edges by W'.
3. Draw diameters on the circle starting from the nodes on it.
4. If an arc is incident with a diameter, set the corresponding edge weight in W to one otherwise set it to zero.

Example

We show the procedure for a 4 nodes ring and distance vector $W' = (2, 3, 2, 1)$



$W' = (2,3,2,1)$

$W = (1,1,1,0)$

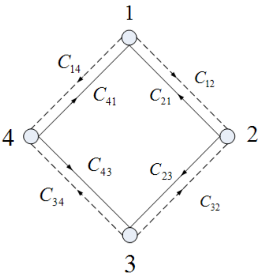
This shows that for instance bound $3R_{1\to 3} + 2R_{2\to 3} + 5R_B \leq 2C_1 + 3C_2 + 2C_3 + C_4$ is redundant given the bound $R_{1\to 3} + R_{2\to 3} + 2R_B \leq C_1 + C_2 + C_3$.
Where $R_B$is the broadcast rate.

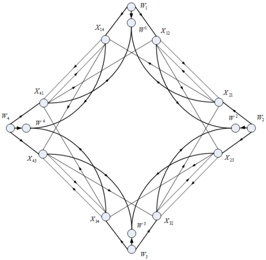
Information Theoretic Upper Bounds

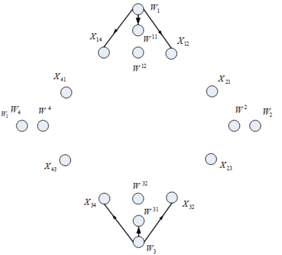
To verify the tightness of routing solution in network coding setting we find upper bounds on network coding capacity of the rings.

- Need to strengthen a generalization of PdE bounds to account for *common* information terms.
- We need to only prove the bounds for all ones distance vectors. For general vectors, the *contraction* of zero weight edges will not affect the bound.

PdE for 4 vertices ring







- *The PdE bound:*

$$R_1 + R_3 \leq C_{12} + C_{14} + C_{32} + C_{34}.$$

$R_1, R_3$are the sum of rates of all sessions with sources at $1, 3$respectively.

- *Strengthening PdE*

We write PdE as follows:

$$R_1 + R_3 \leq H(X_{12}X_{14}X_{32}X_{34}|W_2W_4).$$

$W_2$is the messages originating at 2, $X_{12}$is the bit-stream from 1 to 2, etc.

$$\begin{aligned} H(X_{12}X_{14}X_{32}X_{34}|W_2W_4) &= \\ H(X_{12}X_{14}X_{32}X_{34}W_2W_4) - H(W_2W_4) &= \\ H(X_{12}X_{32}W_2) + H(X_{14}X_{34}W_4) - I(X_{12}X_{32}W_2; X_{14}X_{34}W_4) - H(W_2W_4) &\leq \\ H(X_{12}) + H(X_{14}) + H(X_{32}) + H(X_{34}) - I(X_{12}X_{32}W_2; X_{14}X_{34}W_4). & \end{aligned}$$

then we get

$$R_1 + R_3 + I(X_{12}X_{32}W_2; X_{14}X_{34}W_4) \leq C_{12} + C_{14} + C_{32} + C_{34}.$$

- The term $I(X_{12}X_{32}W_2; X_{14}X_{34}W_4)$includes the *common information.*
- $X_{12}, X_{32}, W_2$is the information that 2uses for decoding messages
- $X_{14}, X_{34}, W_4$is the information that 4uses for decoding messages
- $I(X_{12}X_{32}W_2; X_{14}X_{34}W_4)$ is larger than the information that both 2and 4 decode, and the information that 2sends for 4and 4sends for 2.

Network Coding bound for larger size of vectors

- We extend the PdE argument and it's strengthening for larger network.
- Define Q_1, Q_2, Q_3, Q_4as 4groups of adjacent vertices.

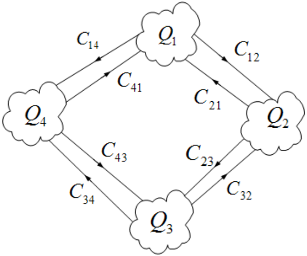


Figure: Four groups of adjacent vertices

The general bound:

$$\sum_{A_i \in U} R_{A_i} + 2 \sum_{A_i \in V} R_{A_i} \leq C_{12} + C_{32} + C_{14} + C_{34}.$$

U and V are the set of sessions with destinations at Q_2or $Q_4$with minimal lengths of 1and 2respectively, with respect to 4ones vector.

Proof of the bounds for two different settings

For general vectors in line setting or in unicast and broadcast setting we have to properly define Q_1, Q_2, Q_3, Q_4.

- If the number of vertices is even, define $Q_1$and Q_3as two opposite nodes, and Q_3, Q_4as the remaining nodes. Sum over all partial bounds to get the corresponding general bound.
- If the number of vertices is odd, define $Q_1$and Q_3as two furthest nodes on the cycle, and Q_3,Q_4as the remaining nodes. Sum over all possible partial cases and divide the result by two to get the corresponding general bound.

Proof hint: It comes from geometries of line, unicast and broadcast sessions and the fact that the minimal trees are preserved by the decomposition process.

General Multiple Multicast

The problem in the general case is very challenging.
The binary distance vectors are not sufficient for routing bounds.

Theorem

For a ring with n nodes the distance vector of $W' = (x, 1, 1, \cdots, 1)$*can not be reduced for* $2 \leq x \leq \frac{n}{2}$

Proof.

Consider set of sessions with nodes $\{1, 2, x+2, x+3, \cdots, n\}, \{1, 2, 3, x+3, \cdots, n\}, \{1, 2, 3, 4, x+4, \cdots, n\}, \cdots, \{1, 2, \cdots, n-x+1\}$. If $W = (W_1, \cdots, W_n)$is the reduction of W', then the minimal routing lengths imply ☐

$$W_2 = W_{x+2}, W_3 = W_{x+3}, \cdots, W_{n-x} = W_n.$$

Next consider sessions with nodes $\{1, 3, x+4, \cdots, n\}, \{1, 3, 4, x+5, \cdots, n\}, \cdots, \{1, 3, 4, \cdots, n-x-1, n\}$and session with nodes $\{1, 3, 4, \cdots, n-x\}$. The minimal routing lengths imply

$$W_2 = W_{x+4}, W_3 = W_{x+5}, \cdots, W_{n-x-1} = W_n.$$

The two relationships imply that $W_2 = W_3 = \cdots = W_n$, and $W_1 = xW_2$. So $W_1 = W_2 = \cdots = W_n = 0$, which is a trivial distance vector.

Conclusions

- We showed that routing rate region can be characterized in terms of relatively few number of inequalities for two multicast problems.
- The ongoing work is to characterize the general multiple multicast routing rate region for ring networks.
- We derived an extension of PdE bound which was tight for our problem of study.
- The ongoing work is to further strengthen the PdE bound to characterize the general multiple multicast network coding capacity.
- Another direction of our work is to find general bounds for general networks on the maximal edge distances that characterize the routing rate region.