Regional Information Capacity of the Linear Time-Varying Channel

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Introduction: Time-invariant and Time-varying

Recall Shannon's main theorem on the information capacity of the linear, time-invariant channel for band-limited signals. Let T_h denote the operator

$$(T_h s)(t) = \int_0^\infty h(\tau)s(t-\tau)d\tau. \tag{1}$$

Shannon proved that the normalized capacity of the channel $r(t) = (T_h)s(t)+n(t)$ for h band-limited to $[-W,W]$ and n AWGN with variance η^2 is

$$\frac{1}{2W}\int_{-W}^{W} \log(1+\frac{|\hat{h}(\omega)|^2}{\eta^2})d\omega. \tag{2}$$

Here we pursue an analogous information capacity result for the time-varying channel. We introduce time-dependency to equation (1) by considering the following channel

$$r(t) = \int h(t,t-\tau)s(\tau)d\tau + n(t). \tag{3}$$

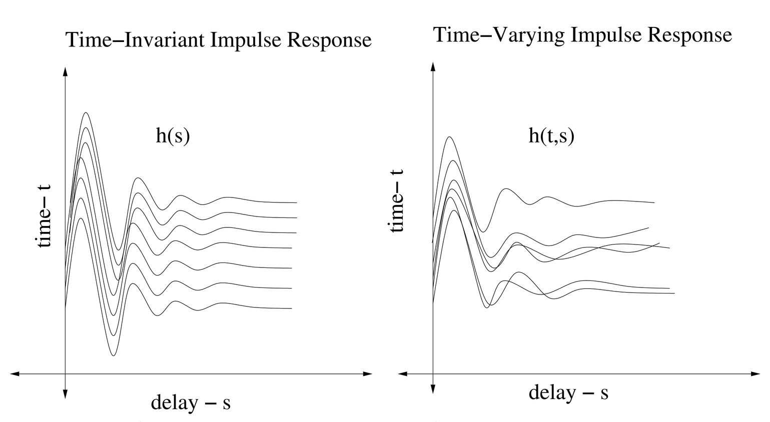

Figure 1: Time-invariant and time-varying impulse responses.

We address the following questions:

- what are the eigenfunctions of the operator in (3)?
- what is an appropriate space of signals for (3)?
- what are the singular values of the channel?
- what, in turn, is the information capacity (maximum mutual information)?

We introduce the time-delay operator $T_x f(t) = f(t-x)$ and the frequency shift operator $M_\omega f(t) = e^{2\pi i\omega t}f(t)$. By defining $\sigma(t,\omega) = \mathcal{F}_2 h(t,\cdot)$ and using several Fourier transforms, we have

$$\int h(t,t-\tau)s(\tau)d\tau = \int\int \hat{\sigma}(\omega,x)M_\omega T_{-x}s(t)d\omega dx. \tag{4}$$

We therefore may equivalently view the channel as a Weyl pseudodifferential operator

$$L_\sigma s(t) = \int\int \hat{\sigma}(\omega,x)e^{-\pi i x\omega}T_{-x}M_\omega s(t)d\omega dx. \tag{5}$$

Our assumption is that the spreading function $\hat{\sigma}$ decays exponentially; that is, there exist constants $\alpha,\beta,C>0$ such that

$$|\hat{\sigma}(\omega,x)| \leq Ce^{-\beta|\omega|-\alpha|x|}. \tag{6}$$

Decay in the second variable of $\hat{\sigma}$ indicates that the time-varying impulse response decays exponentially. Decay in the first variable means that $\sigma(t,\omega)$ evolves smoothly. A large β indicates that the channel is stable, or evolves smoothly, in time, whereas a large α indicates that the channel is stable, or varies smoothly, in frequency. Also, as $\beta\to\infty$ the channel approaches the time-invariant regime.

Main ideas:

- Construct signals with appropriate time-frequency localization that are approximate eigenfunctions of L_σ.
- Estimate the eigenvalues of $\mathbf{A}^*\mathbf{A}$ using approximate diagonalization.
- Eigenvalues are approximately given by samples of $\mathcal{S} = \sigma\sharp\overline{\sigma}$. ($\mathcal{S}$ is explained below.)
- Samples of $\mathcal{S}$ are analogous to samples of $|\hat{h}(\omega)|^2$ in the time-invariant case.
- Results hold for a *class* of channels, i.e. not dependent on specific channel.
- Construction is *adaptive* for different channel classes, i.e. different stability parameters α,β.

Signaling Set and Approximate Diagonalization

The basic approach of our work is to design a system that can adapt to the parameters α and β to exploit either time or frequency stability. The foundation of the results that follow is the fact that orthonormal signaling systems with controllable time-frequency localization do exist. In particular, one may construct a function ψ_s with the following time-frequency localization

$$|\psi_s(t)| \leq Ce^{-D\frac{\pi}{s}|t|} \quad \forall\, t\in\mathbb{R}$$
$$|\widehat{\psi_s}(\omega)| \leq Ce^{-D\pi s|\omega|} \quad \forall\, \omega\in\mathbb{R},$$

where $0<D<1$ is a positive constant, such that $\{\psi_{kl}\}_{k,l\in\mathbb{Z}} = \{M_{\rho bl}T_{\rho ak}\psi_s\}_{k,l\in\mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R})$ for $ab=1$ and $\rho>1$. We set $a=\frac{\beta}{\alpha}$, $b=\frac{\alpha}{\beta}$ and $s=(\frac{\beta}{\alpha})^2$. Thus, when the channel is highly stable in time we use signals that are long in time, and localized in frequency. Additionally, these signals will have a small separation in frequency. Conversely, when the channel is stable in frequency we use signals localized in time.

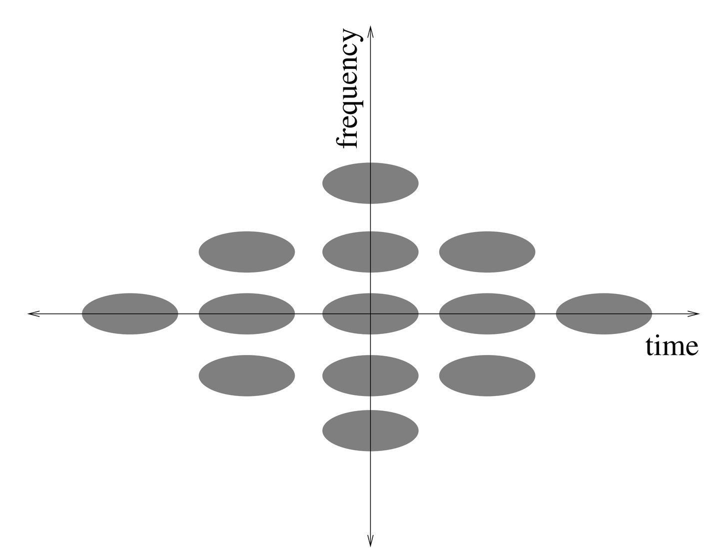

Figure 2: Signal Time-frequency localization for a slightly time-stable channel.

We assume that α and β are known to both the transmitter and receiver. Given a region $R=[0,T]\times[-W,W]$ of the time-frequency plane, the transmitter uses the following regime:

P1) ψ has the properties described above and is constructed according to Proposition 3.5 in [FS08].

P2) the signals are contained in the finite dimensional space

$$\Psi = \text{span}\{\psi_{k,l}\}_{k=0,l=-L}^{K,L}$$
$$= \text{span}\{M_{\rho\frac{\alpha}{\beta}l}T_{\rho\frac{\beta}{\alpha}k}\psi\}_{k=0,l=-L}^{K,L}.$$

P3) $K=\lfloor\frac{T}{\rho}\frac{\alpha}{\beta}\rfloor$ and $L=\lfloor\frac{2W}{\rho}\frac{\beta}{\alpha}\rfloor$.

The main properties of this system are that the signals are *orthonormal* and the *signals are approximate eigenfunctions* of the channel (3).
We may now view the channel as a map $L_\sigma:\Psi\to L^2(\mathbb{R})$. We express this map as a matrix by setting

$$\mathbf{A}_{klk'l'} = \langle L_\sigma\psi_{k',l'},\varphi_{kl}\rangle, \quad -K\leq k\leq K,\ 0\leq l'\leq L, \quad k,l\in\mathbb{Z}, \tag{7}$$

where $\varphi_{kl} = M_{\frac{1}{\rho}\frac{\alpha}{\beta}l}T_{\frac{1}{\rho}\frac{\beta}{\alpha}k}\psi_s$. The receiving set $\{\varphi_{kl}\}_{k,l\in\mathbb{Z}}$ spans $L^2(\mathbb{R})$ (an important but subtle point). When $\{\varphi_{kl}\}_{k,l\in\mathbb{Z}}$ is used at the receiver, the noise samples will not be uncorrelated. Nonetheless, due to the frame properties of the receiver, one still has that the normalized information capacity of the system is

$$\frac{1}{|R|}\sum_{k=-K,l=0}^{K,L}\log\left(1+\frac{\lambda_{k,l}(\mathbf{A}^*\mathbf{A})}{\eta^2}\right). \tag{8}$$

See also [MG95, Dig97, BS99, DBS07] for similar approaches.

Explanation of $\mathcal{S}$: the composition of L_σ and L_τ is given by $L_\sigma L_\tau = L_{\sigma\sharp\tau}$, where $\sigma\sharp\tau = \mathcal{F}^{-1}(\hat{\sigma}\natural\hat{\tau})$ is called the *twisted product* of σ and τ, and

$$(\hat{\sigma}\natural\hat{\tau})(\omega,x) = \iint \hat{\sigma}(\omega',x')\hat{\tau}(\omega-\omega',x-x')e^{-\pi i(x\omega'-\omega x')}d\omega' dx'$$

is called the *twisted convolution* of $\hat{\sigma}$ and $\hat{\tau}$. Also, $L_\sigma^* = L_{\overline{\sigma}}$, and we set $\mathcal{S}^+(x,\omega) = max(0,S(x,\omega))$.

Main Theorems

Theorem 1.1 (Information Capacity for CSIR) *Let R be a closed rectangular region of the time-frequency plane. Assume the channel is given by $r=L_\sigma s+n$, $n\sim\mathcal{N}(0,\eta^2)$, that $\hat{\sigma}$ satisfies the decay condition (6), and that the receiver has perfect channel knowledge, while the transmitter knows only α, β and R. Also assume the signaling set is given according to properties P1-P3 above. Set $\mathcal{S}=\overline{\sigma}\sharp\sigma$ and $\mathcal{S}^+(x,\omega)=(\mathcal{S}(x,\omega))^+$. Then*

$$\left|\mathcal{IC}_{\sigma,R} - \sum_{k=0,l=-L}^{K,L}\log\left(1+\frac{\mathcal{S}^+(\rho\frac{\beta}{\alpha}k,\rho\frac{\alpha}{\beta}l)}{\eta^2}\right)\right| \leq 2KL\log(1+\mathcal{O}(e^{-\frac{\rho}{4}(\beta+\alpha)}+\frac{1}{(\alpha\beta D)^2})) \tag{9}$$

Theorem 1.2 (Information Capacity for CSIT) *In addition to the assumptions of Theorem 1.1, assume now that both the transmitter and receiver have perfect channel knowledge. Let P be the total power available to the transmitter. Then*

$$\left|\mathcal{IC}^w_{\sigma,R} - \sum_{k=0,l=-L}^{K,L}\log\left(1+\frac{P^S_{kl}\mathcal{S}^+(\rho\frac{\beta}{\alpha}k,\rho\frac{\alpha}{\beta}l)}{\eta^2}\right)\right| \leq 2KL\log(1+2KL\cdot\mathcal{O}(e^{-\frac{\rho}{4}(\beta+\alpha)}+\frac{1}{(\alpha\beta D)^2})), \tag{10}$$

where, $\{P^S_{kl}\}_{k=0,l=-L}^{K,L}$, $\sum_{k=0,l=-L}^{K,L}P^S_{kl}=P$, denotes the water-filling allocation based on the approximate eigenvalues $\{\mathcal{S}^+(\rho\frac{\beta}{\alpha}k,\rho\frac{\alpha}{\beta}l)\}_{k=0,l=-L}^{K,L}$.

We now consider a sequence of time-varying channels approaching the time-invariant channel. We assume the sequence of Weyl symbols has the following properties:

P4) $|\hat{\sigma}_n(\omega,x)|\leq C_n e^{-\beta_n|\omega|-\alpha|x|}$. Note that α is fixed for all $n\in\mathbb{N}$.

P5) there exist positive constants M and m such that

$$M \;\geq\; \int\int|\hat{\sigma}(\omega,x)|d\omega dx \;\geq\; m \quad \forall\, n\in\mathbb{N}.$$

P6) $\beta_n>\beta_{n'}$ for $n>n'$ and $\lim_{n\to\infty}\beta_n=\infty$.

Theorem 1.3 (Time-Invariant Capacity for CSIR) *Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be a sequence of Weyl symbols converging to the linear time-invariant channel given by the impulse response $h(x)$ according to properties P4-P6. Set $R_n=[0,\frac{\beta_n}{\alpha}]\times[-W,W]$, and let the noise and signaling system be as in Theorem 1.1. Then*

$$\lim_{n\to\infty}\frac{1}{|R_n|}\sum_{k=0,l=-L}^{K,L}\log\left(1+\frac{\mathcal{S}^+(\rho\frac{\beta}{\alpha}k,\rho\frac{\alpha}{\beta}l)}{\eta^2}\right) = \frac{1}{2\rho W}\int_{-W}^{W}\log\left(1+\frac{|\hat{h}(\omega)|^2}{\eta^2}\right)d\omega. \tag{11}$$

Theorem 1.4 (Time-Invariant Capacity for CSIT) *We make the same hypotheses and definitions as for Theorem 1.3, except that the transmitter now also has perfect channel knowledge and total power resources P_{Total}. Let $\{P^{S_n}_{kl}\}_{k=0,l=-L}^{K,L}$ denote the water-filling power allocations based on the approximate eigenvalues $\mathcal{S}^+_n(\rho\frac{\beta}{\alpha}k,\rho\frac{\alpha}{\beta}l)$. Then*

$$\lim_{n\to\infty}\frac{1}{|R_n|}\sum_{k=0,l=-L}^{K,L}\log\left(1+\frac{P^{S_n}_{kl}\mathcal{S}^+(\rho\frac{\beta}{\alpha}k,\rho\frac{\alpha}{\beta}l)}{\eta^2}\right) = \frac{1}{2\rho W}\int_{-W}^{W}\log\left(1+\frac{P(\omega)|\hat{h}(\omega)|^2}{\eta^2}\right)d\omega, \tag{12}$$

where $P(\omega)$ is the power allocation determined by water-filling, as in the classical time-invariant case, and $\int_{-W}^{W}P(\omega)d\omega=P_{Total}$.

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