Noisy Feedback Reliability of a Peak Power Constrained AWGN Channel

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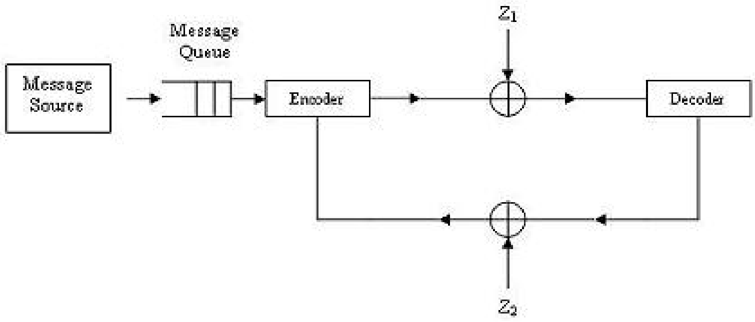
The Communication System We are Concerned With



Key features:

- Source produces M-ary messages regularly.
- Messages pile up in a queue.
- Encoder transmits them to the Decoder over the channel.

The Forward AWGN Channel

- Specified by
$$Y(t) = X(t) + Z_1(t)$$
where $X(t)$ is the input signal, $Z_1(t)$ is a white noise signal having one-sided power spectral density N_1 and $Y(t)$ is the resulting output.
- The input is restricted to having an average power P_1 and a peak power that must equal the average power. This is the tightest peak power constraint.
- No restriction on the bandwidth used by the input signal.

The Feedback AWGN Channel

- Specified by
$$W(t) = V(t) + Z_2(t)$$
where $V(t)$ is the input signal, $Z_2(t)$ is a white noise signal having one-sided power spectral density N_2 and $W(t)$ is the resulting output.
- $Z_2(t)$ is independent of forward channel noise $Z_1(t)$.
- The input is restricted to having an average power P_2 and a peak power that must equal the average power.
- No restriction on the bandwidth used by the input signal.
- Channel capacity is the ratio $C_2 = \frac{P_2}{N_2}$.

Communicating Messages over the Forward AWGN Channel in the Presence of the Feedback AWGN Channel

A Step-by-Step Approach

Motivated by work of Sahai and Simcek on Discrete Memoryless Channels.

1. Transmit a message over the forward channel.
2. Get feedback about how it was decoded.
3. Transmit Confirm/Deny signal.
4. Get feedback about how it was decoded.
5. Retransmit message or transmit new message.

Encoder Phase 1

Transmit a message over the forward channel

- Message m, $m \in \{1, \cdots, M\}$, arrives from queue at time $t = 0$.
- Transmit $s_m(t)$, $0 \leq t \leq T_1$, from orthogonal signal set.
- At time $t = T_1$ Decoder performs Maximum Likelihood decoding.
- Resulting error exponent is

$$E_{orth}(R) = \begin{cases} C_1/2 - R, & R \leq C_1/4 \\ (\sqrt{C_1} - \sqrt{R})^2, & C_1/4 \leq R \leq C_1 \end{cases}$$

Decoder Phase 1

Get feedback about how transmitted message is decoded

- Decoder's estimate of transmitted message is $m^{'}$ where $m^{'} \in \{1, \cdots, M\}$.
- Transmit $b_{m^{'}}(t)$, $0 \leq t \leq T_3$, from Decoder's orthogonal signal set, at time $t = T_1$ over feedback channel.
- Encoder only interested in knowing if $b_m(t)$ was transmitted.
- This is an Identification problem. We follow the decision rule of Burnashev.
- At time $t = T_1 + T_3$, Encoder projects received signal onto $b_m(t)$.
- If value of projection onto $b_m(t)$ is greater than z where $z > 0$, Encoder decides that $b_m(t)$ was transmitted.

Decoder Phase 1

Analysis

- Do a False Alarm/Missed Detection type of analysis.
- Fixing Missed Detection probability to be ϵ_1, bound the False Alarm probability as

$$P_{fa}^{(1)} \leq \frac{1}{2} e^{-\frac{1}{2}(x^*)^2} e^{-\left(C_2 T_3 - \sqrt{2 C_2 T_3} x^*\right)} \text{ for } T_3 > \frac{(x^*)^2}{2 C_2}$$

where x^* is a constant.

Encoder Phase 2

Transmit Confirm/Deny signal

- At time $t = T_1 + T_3$, Encoder transmits a Confirm if it believes $m^{'} = m$ and a Deny otherwise.
- A pair of antipodal signals $\{r(t), -r(t)\}$, $0 \leq t \leq T_2$, satisfying peak and average power constraints, is used to indicate Confirm and Deny, respectively.
- At time $t = T_1 + T_3 + T_2$, the Decoder projects the received signal onto $r(t)$.
- If the value of the projection onto $r(t)$ is greater than γ where $\gamma > 0$, Decoder decides that a Confirm was transmitted. In this case it finalizes $m^{'}$.
- Otherwise, the Decoder decides that a Deny was transmitted and discards $m^{'}$.

Encoder Phase 2

Analysis

- We do a False Alarm/Missed Detection type of analysis.
- Fixing Missed Detection probability to be ϵ_2, we bound the False Alarm probability as

$$P_{fa}^{(2)} \leq \frac{1}{2} e^{-\frac{1}{2}(y^*)^2} e^{-(4C_1T_2 - 2y^*\sqrt{2C_1T_2})} \text{ for } T_2 > \frac{(y^*)^2}{8C_1}$$

where y^* is a constant.

Decoder Phase 2

Get feedback about how Confirm/Deny signal was decoded.

- At time $t = T_1 + T_3 + T_2$, if Decoder decoded a Confirm, its state is set to 1 ($m^{'}$ is finalized).
- If Decoder decoded a Deny, its state is set to 0 ($m^{'}$ is discarded).
- Encoder moves on to transmitting the next message, from $t = T_1 + T_3 + T_2$ to $t = 2(T_1 + T_3 + T_2)$ and so on.
- Over time, as a stream of messages is received, the Decoder will have a state sequence consisting of 0's and 1's.
- An *Anytime* code with ML decoding is used to transmit this state sequence to the Encoder over the feedback channel.
- The resulting probability of error is

$$P_{dp2} < K^{'} e^{-WT_4 E_{orth}(\frac{1}{T_4})}$$

where $K^{'}$ is a positive constant, W is a code parameter, and E_{orth} is the orthogonal signalling error exponent of the feedback channel.

Probability of Uncorrectable Error

Finalizing an Incorrect Message

- The possible error events are:
  1. A Confirm is sent when a Deny should have been sent. This has exponent $C_2 T_3$.
  2. A Deny is decoded as a Confirm. This has exponent $4C_1 T_2$
  3. The wrong message is transmitted due to error in the Anytime code. This has exponent $WT_4 E_{orth}(R)$
- Choose T_3 and W to equate exponents to $4C_1 T_2$.
- Overall probability of error is the probability of the union of these three events.
- Using the Union Bound we get the overall exponent $4C_1 T_2$.

Expected Time to Finalize a Message

- Retransmission of a message is required if
  1. The message is incorrectly received. This has probability
  $$P_e^{(1)} = f(R, T_1)e^{-T_1 E_{orth}(R)}$$
  2. A Deny is sent when a Confirm should have been sent. This has probability ϵ_1
  3. A Confirm is decoded as a Deny. This has probability ϵ_2
- Thus the probability that a message will be accepted is lower bounded by
$$1 - (P_e^{(1)} + \epsilon_1 + \epsilon_2)$$
- Possible times of accepting a message are $\{(T_1 + T_3 + T_2), (W + 4)(T_1 + T_3 + T_2), (2W + 6)(T_1 + T_3 + T_2), (3W + 8)(T_1 + T_3 + T_2), \cdots\}$

Expected Time to Finalize a Message

Analysis

- Compare with a geometric random variable.
- Thus, we obtain that the expected time for a message to be accepted, $\bar{\tau}$, is approximately upper bounded by

$$T_1 + T_2 \left(1 + \frac{4C_1}{C_2} \right)$$

Bound on the Error Exponent

- Start with

$$\bar{\tau} < T_1 + T_2 \left(1 + \frac{4C_1}{C_2}\right)$$

- After some manipulation we obtain

$$E(\bar{R}) > \left(\frac{1}{C_2} + \frac{1}{4C_1}\right)^{-1} \left(1 - \frac{\bar{R}}{C_1}\right)$$

- As C_2 grows, lower bound approaches

$$4(C_1 - \bar{R})$$

- This is the error exponent in the presence of perfect (noiseless) feedback, derived by Schalkwijk and Barron in 1971.